Discrete Optimization: Case Studies From Real Life
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Optimal Wire Ordering and Spacing in Low Power Semiconductor Design

Dedicated to:
Richie Sambora, Bon Jovi - "Blood On Blood"
Cliff Burton, Metallica - "Creeping Death"
To my deceased bunny Willy the Second, I miss you dearly

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1 Introduction

Today one can find low power semiconductor chips in nearly every electronic integrated machine, e.g. mobile phones, cars,... The most famous example is the CPU (central prosessing unit) used in home PCs.

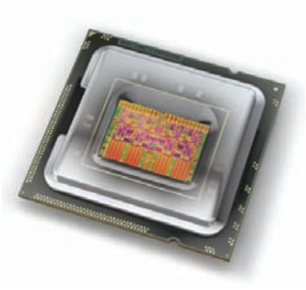

Fig. 1 CPU: Intel Core i7 *Courtesy of Intel Corp.*

On these semiconductor chips there are a couple of wires realising integrated circuits. The total length of all wires can rise up to some kilometres. Caused by construction demands the wires are rectangular and parallel to each other. One chip consists of up to ten layers, one superimposed on the other, of these orderd wires. This structure is the reason for electromagnetical fields that are generated between the wires. This fields are causing enery and thermal losses.

Now the optimization problem can be thought of as the task to minimize the forfeitures by finding the optimal wire placement (OWP). For OWP there are two possibilities of optimization, the wire ordering (OWO) and the wire spacing (OWS), without modifying the boundary conditions, e.g. chip size.

2 Physical Properties

It is clear that we only can diversify the distances between wires and their order on a layer. Other parameters, like materials for constructions or the voltage used in the circuits, are predetermined by construction design and guidelines. Also the problem has to be restricted to only one layer, because the distances and the order of the layers are also fixed.

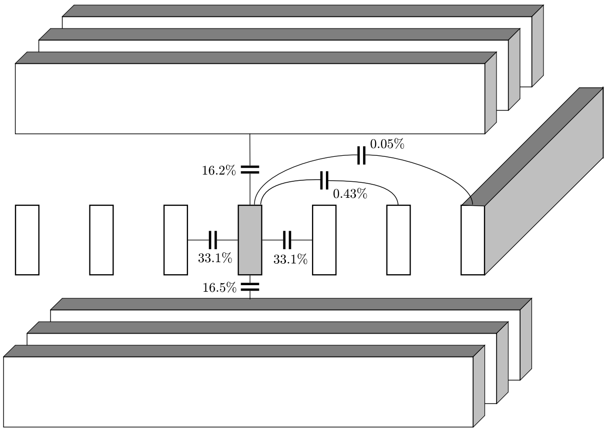

Fig. 2 prozentual influence of the fields

First of all it is essential to clarify the problem in order to carry out appropriate physical models.
It seems to be very reasonable to split up the whole problem into three partial problems Ⓐ, Ⓑ and Ⓒ.
Ⓐ is dealing with the fields generated between the side faces of the wire i and its neighbors $i-1$ and $i+1$. Ⓑ covers fields produced between the ground and top faces of the wire i and its neighbors $i-1$ and $i+1$. That is very similar to Ⓒ, covering the fields between ground and top faces of the wire i and its neighbors $i-2$ and $i+2$.
Except this mentioned fields, all other influences shall be disregarded, because their ascendancies do not substantially contribute to energetical and thermal losses.
Now we want to put the losses for each of the partial problems successively in a physical formula and at the end assemble the partial formulas to one entire model.

Ad Ⓐ:

The field energy of a capacitor is of outstanding importance for the energetical and thermal losses, because of its equivalence to the dissipated energy in our problem. This energy can be specified as:

$$E = \frac{1}{2} \cdot C \cdot U^2 = \frac{1}{2} \cdot \varepsilon \cdot \frac{A}{d} \cdot U^2$$

where C is the capacitance, U the voltage, A the surface and ε the permittivity of the capacitor plates. The distance between the plates is given by d.

Architectonically conditioned all parameters except the distance of the wires are fixed. So only the distances between the wires seem to be crucial. This leads to the following physical model for losses in wire w:

$$E_{Loss\textcircled{A}} = \alpha_{\textcircled{A}}(w) \cdot \left(\frac{1}{x_{left}} + \frac{1}{x_{right}} \right)$$

with x_{left} the distance to the left neighbor wire and x_{right} the distance to the right neighbor wire. In $\alpha_{\textcircled{A}}$ all constant factors are included.

Ad Ⓑ:

In analogy to Ⓐ all parameters except the wire distances are fixed. The wires are very slim and can in the following be considered as negligible slim wires. Their altitude is in comparison to their width considerably larger. The field energy can again be depicted as:

$$E = \frac{1}{2} \cdot C \cdot U^2 = \frac{1}{2} \cdot \varepsilon \cdot \frac{A}{d} \cdot U^2$$

In this case anon the distance between the capacitor plates is decisive. Involving the experimental findings for field line distribution and the negligibility of the wire width enable to model the distance as the length of the hemicycle from centre to centre of the ground faces, and analogous for the top faces, between a wire i and its neighbor $i+1$ and $i-1$. This is resulting in a similar representation of the energy losses as in Ⓐ:

$$E_{Loss\textcircled{B}} = \alpha_{\textcircled{B}}(w) \cdot \left(\frac{1}{x_{left}} + \frac{1}{x_{right}} \right)$$

Again all constant factors are resumed in $\alpha_{\textcircled{B}}$ in analogy to Ⓐ. It has to be regarded that $E_{Loss\textcircled{B}}$ is very diminutive in comparison to $E_{Loss\textcircled{A}}$, because the surface of the capacitor plates in Ⓑ are very diminutive in comparison to the one in Ⓑ. Also the distances are largish.

Ad Ⓒ:

This problem can be modelled in analogy to Ⓑ. As capacitor plate distance here we have to pick again the length of the hemicycle from centre to centre of the ground faces, and analogous for the top faces, between a wire i and its neighbor. But in this case we take the neighbors $i+2$ and $i-2$. The function for the energy losses appears as:

$$E_{Loss\textcircled{C}} = \alpha_{\textcircled{C}}(w) \cdot \left(\frac{1}{x_{left_1} + x_{left_2}} + \frac{1}{x_{right_1} + x_{right_2}} \right)$$

The constant factors again are summarised in $\alpha_{\textcircled{C}}$, x_{left_1} is the distance to the first left neighbor wire and x_{left_2} the distance between first and second wire. Analogous for x_{right_1} and x_{right_2}.

Here again it has to be regarded that $E_{Loss\textcircled{C}}$ is obviously lower than $E_{Loss\textcircled{B}}$, because of the larger distances.

Ad Summa:

Recapitulatory the total energy losses are the superposition of the partial losses in Ⓐ, Ⓑ and Ⓒ:

$$\begin{aligned}
E_{Loss_{total}} &= E_{Loss\textcircled{A}} + E_{Loss\textcircled{B}} + E_{Loss\textcircled{C}} \\
&= \alpha_{\textcircled{A}}\,(w) \cdot \left(\frac{1}{x_{left}} + \frac{1}{x_{right}}\right) \\
&+ \alpha_{\textcircled{B}}\,(w) \cdot \left(\frac{1}{x_{left}} + \frac{1}{x_{right}}\right) \\
&+ \alpha_{\textcircled{C}}\,(w) \cdot \left(\frac{1}{x_{left_1} + x_{left_2}} + \frac{1}{x_{right_1} + x_{right_2}}\right)
\end{aligned}$$

$E_{Loss\textcircled{A}}$ and $E_{Loss\textcircled{B}}$ can further be combined to

$$E_{Loss\textcircled{A}+\mathrm{B}} = \alpha_{\textcircled{A}}\,(w) \cdot \left(\frac{1}{x_{left}} + \frac{1}{x_{right}}\right)^{1+\varepsilon}, \text{small } \varepsilon > 0$$

This is possible, because $E_{Loss\textcircled{B}}$ arises from the direct neighbors of the wire. It is also relativly small in comparision to $E_{Loss\textcircled{A}}$. This property leads to following total equation:

$$\begin{aligned}
E_V &= E_{Loss\textcircled{A}+\textcircled{B}} + E_{Loss\textcircled{C}} \\
&= \alpha_{\textcircled{A}}\,(w) \cdot \left(\frac{1}{x_{left}} + \frac{1}{x_{right}}\right)^{1+\varepsilon} \\
&+ \alpha_{\textcircled{C}}\,(w) \cdot \left(\frac{1}{x_{left_1} + x_{left_2}} + \frac{1}{x_{right_1} + x_{right_2}}\right)
\end{aligned}$$

Notice that $E_{Loss\textcircled{C}}$ is infinitesimal in comparision to $E_{Loss\textcircled{A}+\textcircled{B}}$.

We consider a scenario involving N parallel wires which are regarded as being enclosed between two static wires with switching frequencies 0. On a chip these boundary wires could be power or shield wires.

In the following let $N \in \mathbb{N}$, and let $w_1, \ldots, w_N$ denote different *(proper) parallel wires*. Further, let w_0 and w_{N+1} be two additional *'dummy wires'*, and set $W = \{w_1, \ldots, w_N\}$ and $\widehat{W} = W \cup \{w_0, w_{N+1}\}$.

Let $r \in]0,\infty[$ be the given *spacing range*, and let $d \in]0,r]$ be the *minimum accepted inter wire distance*. Then, a *wire placement* is a map $\varphi : \widehat{W} \to [0,r]$ such that $\varphi(w_0) = 0$, $\varphi(w_{N+1}) \geq \varphi(w)$ for $w \in W$ and

$$|\varphi(w) - \varphi(w')| \geq d$$

for $w, w' \in \widehat{W}$ with $w \neq w'$. As it turns out, the underlying optimization problem can be described best in terms of the two separate tasks of wire ordering and wire spacing.

A *wire ordering* is a bijection $\pi : \widehat{W} \to \{0, 1, \ldots, N, N+1\}$ such that $\pi(w_0) = 0$ and $\pi(w_{N+1}) = N+1$. Let $\mathcal{P}_N$ denote the set of all wire orderings for a given number N. An admissible *wire spacing* is a function $\delta : \{0, 1, \ldots, N, N+1\} \to [0,r]$ with $\delta(0) = 0$ and

$$\delta(j) + d \leq \delta(k)$$

for any $j, k \in \{0, 1, \ldots, N+1\}$ with $j < k$. Let $\mathcal{D}_N(r,d)$ denote the set of all admissible wire spacings. Note that the above constraints are already implied by the conditions on all pairs of adjacent positions.

Of course, any pair (π, δ) of a wire ordering and a wire spacing constitutes a wire placement φ via $\varphi = \delta \circ \pi$ and vice versa. Hence we will not distinguish between them and, in particular, also speak of (π, δ) as a wire placement.

Finally, let $\alpha_{\textcircled{A}} : W \to [0,\infty[$ encode the *switching frequencies* of the proper wires in problem Ⓐ, analogous for $\alpha_{\textcircled{B}} : W \to [0,\infty[$ and $\alpha_{\textcircled{C}} : W \to [0,\infty[$. The set of all such functions will be denoted by $\mathcal{A}_N$.

Then, the *power loss* $L(\pi, \delta)$ of a wire placement (π, δ) is given by

$$\begin{aligned} L(\pi,\delta) = {} & \sum_{w \in W} \alpha_{\textcircled{A}}(w) \left(\frac{1}{\delta(\pi(w)) - \delta(\pi(w)-1)} + \frac{1}{\delta(\pi(w)+1) - \delta(\pi(w))} \right)^{1+\varepsilon} \\ & + \sum_{w \in W} \alpha_{\textcircled{C}}(w) \left(\frac{1}{\delta(\pi(w)) - \delta(\pi(w)-2)} + \frac{1}{\delta(\pi(w)+2) - \delta(\pi(w))} \right), \end{aligned}$$

and the *optimal wire placement* problem (OWP) is the following task: Given $N \in \mathbb{N}$, $r, d \in]0,\infty[$, and $\alpha \in \mathcal{A}_N$, find $\pi^* \in \mathcal{P}_N$ and $\delta^* \in \mathcal{D}_N(r,d)$ such that

$$L(\pi^*, \delta^*) = \min \{ L(\pi, \delta) : \pi \in \mathcal{P}_N \ \wedge \ \delta \in \mathcal{D}_N(r,d) \},$$

or decide that no such minimum exists.

Note that the specific set W does not play any role. All that matters are the switching frequencies associated with the wires. Also the function π can be identified with a permutation on $\{1, \ldots, N\}$. The concise mathematical formulation of OWP given in the next section uses this abstraction and describes the task in terms of the variables x_i for $i = 1, \ldots, N+1$, that are related to the functions π and δ through

$$x_{\pi(w)} = \delta(\pi(w)) - \delta(\pi(w) - 1).$$

The switching frequencies will be encoded by a vector $(s_1, \ldots, s_N)$ where $s_i = \alpha(w_i)$ for $i = 1, \ldots, N$.

3 The Main Result

Let $\mathcal{S}_N$ denote the *symmetric group* on N elements. We are dealing with the following mathematical optimization problem.

Problem: Optimal Wire Placement (OWP)
Instance: $N \in \mathbb{N}$; $s_1, \ldots, s_N \in [0, \infty[$; $d, r \in]0, \infty[$.
Question: Decide whether there exists a solution (π, x) of

$$\begin{aligned} \min \sum_{i=1}^{N} \Bigg(& s_{\pi(i)} \left(\frac{1}{x_i} + \frac{1}{x_{i+1}} \right)^{1+\varepsilon} \\ & + s_{\pi(i)} \left(\frac{1}{x_{i-1} + x_i} + \frac{1}{x_{i+1} + x_{i+2}} \right) \Bigg) \\ \text{s. th.} \quad & \sum_{i=1}^{N+1} x_i \le r \\ & x_i \ge d \quad (i = 1, \ldots, N+1) \\ & \pi \in \mathcal{S}_N, \end{aligned}$$

and, if so, give one.

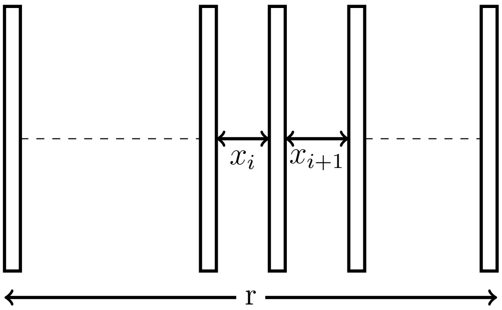

Fig. 3 Main Problem

When the permutation π is fixed, we are confronted with an instance of Optimal Wire Spacing (OWS); the input is the same but the objective is just to find optimal wire distances x_i.

4 Optimal Wire Spacing: Field Ⓐ and Ⓑ

Problem: Optimal Wire Spacing (OWS)
Instance: $N \in \mathbb{N}$; $q_1, \dots, q_{N+1} \in [0, \infty[$; $d, r \in]0, \infty[$, $\varepsilon > 0$.
Question: Decide whether there exists a solution of

$$\begin{aligned} \min \sum_{i=1}^{N+1} \frac{q_i}{x_i^{1+\varepsilon}} & \\ \text{s. th.} \sum_{i=1}^{N+1} x_i \leq r & \\ x_i \geq d (i = 1, ..., N+1) & \end{aligned}$$

and, if so, give one.

By setting $\varepsilon > 0$ we get the solution for the problem which only considers field 1.
To find a solution for this problem there are two ways. First we compute a feasible and optimal solution by using the Karush-Kuhn-Tucker Theorem. Subsequently we give an alternative solution applying the Hölder's inequality.

Solution with Karush-Kuhn-Tucker Theorem:

The following lemma characterizes optimal wire spacings.

Lemma 1 *Let $(N, q_1, \dots, q_{N+1}, r, d)$ be an instance of OWS with $r > (N+1)d$, and $q_1, \dots, q_{N+1} > 0$. Then the objective function F is strictly convex on the feasible region P, and the minimum of F over P is uniquely determined.*
For $x = (x_1, \dots, x_{N+1})^T$ define

$$D(x) = \big\{ i \in \{1, \dots, N+1\} : x_i = d \big\} \quad \text{and} \quad R(x) = \{1, \dots, N+1\} \setminus D.$$

Then a vector $x^* = (x_1^*, \dots, x_{N+1}^*)^T$ is the optimal solution, if and only if

$$d < x_k^* = \frac{\sqrt[2+\varepsilon]{q_k}\big(r - |D(x^*)|d\big)}{\sum_{i \in R(x^*)} \sqrt[2+\varepsilon]{q_i}} \leq \frac{\sqrt[2+\varepsilon]{q_k}}{\sqrt[2+\varepsilon]{q_j}} d \tag{1}$$

for all $j \in D(x^*)$ and $k \in R(x^*)$.

Proof To determine a feasible Minimum we now have a closer look at the objective function $F(x) = \sum_{i=1}^{N+1} \frac{q_i}{{x_i}^{1+\varepsilon}}$, which is, of course, differentiable for all feasible points P.

Proof of convexity:
Compute the partial derivation:

$$\frac{\partial F}{\partial x_i} = -(1+\varepsilon)\frac{q_i}{x_i^{2+\varepsilon}}$$
$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \begin{cases} (\varepsilon^2+3\varepsilon+2)\dfrac{q_i}{x_i^{3+\varepsilon}} & \text{for } i=j; \\ 0 & \text{else.} \end{cases}$$

with $i, j \in \{1, ..., N+1\}$

Due to the fact that the Hessian Matrix is positive definite we can infer that F(x) is strictly convex on the feasible region and its minimum is unique. By the Karush-Kuhn-Tucker-Theorem a feasible vektor $x^* = (x_1^*, \ldots, x_{N+1}^*)^T$ is optimal only if there exist non negative Lagrange multipliers $\lambda_0, \ldots, \lambda_{N+1}$ such that

$$(1+\varepsilon)\frac{q_i}{(x_i^*)^{2+\varepsilon}} = -\nabla F(x^*)^T u_i = \lambda_0 - \lambda_i$$

$$\lambda_0 \left(r - \mathbb{1}^T x^*\right) = 0$$

$$\lambda_i \left(x_i^* - d\right) = 0$$

with $i, j \in \{1, ..., N+1\}$

Let x^* be the optimal solution. Since all q_i's are positive, we have $\lambda_0 > 0$, and hence
$$\sum_{i=1}^{N+1} x_i^* = r.$$
As a consequence, $r > (N+1)d$ implies that $R(x^*) \neq \emptyset$.

Now, let $i, k \in R(x^*)$. Then $x_i^*, x_k^* > d$, hence $\lambda_i = \lambda_k = 0$ (follows from (3.4)) and therefore with (3.2):

$$0 = \lambda_i = \lambda_0 - (1+\varepsilon)\frac{q_i}{(x_i^*)^{2+\varepsilon}} = \lambda_0 - (1+\varepsilon)\frac{q_k}{(x_k^*)^{2+\varepsilon}} = \lambda_k$$

$$\text{thus} \qquad (x_i^*)^{2+\varepsilon} = \frac{q_i}{q_k}\,(x_k^*)^{2+\varepsilon},$$

which yields

$$r = \sum_{i=1}^{N+1} x_i^* = \sum_{i \in D(x^*)} x_i^* + \sum_{i \in R(x^*)} x_i^* = |D(x^*)|d + \frac{x_k^*}{\sqrt[2+\varepsilon]{q_k}} \sum_{i \in R(x^*)} \sqrt[2+\varepsilon]{q_i},$$

proving the first part of (1).

On the other hand, for $j \in D(x^*)$ and $k \in R(x^*)$ we get $\lambda_k = 0$ and $\lambda_j \geq 0$, which yields

$$0 = \lambda_k = \lambda_0 - (1+\varepsilon)\frac{q_k}{(x_k^*)^{2+\varepsilon}} \leq \lambda_j = \lambda_0 - (1+\varepsilon)\frac{q_j}{(x_j^*)^{2+\varepsilon}}$$

$$\text{thus} \qquad \frac{q_j}{(x_j^*)^{2+\varepsilon}} = \frac{q_j}{d^2} \leq \frac{q_k}{(x_k^*)^{2+\varepsilon}},$$

completing the 'only if' part of the proof.

Now, let x^* satisfy (1). Then x^* is feasible since $x_i^* \geq d$ for all $i \in D(x^*) \cup R(x^*)$ and

$$\sum_{i=1}^{N+1} x_i^* = |D(x^*)|d + \sum_{j \in R(x^*)} \sqrt[2+\varepsilon]{q_j} \left(\sum_{i \in R(x^*)} \sqrt[2+\varepsilon]{q_i} \right)^{-1} (r - |D(x^*)|d) = r.$$

Further, we have

$$\frac{q_j}{(x_j^*)^{2+\varepsilon}} = \frac{q_j}{d^2} \leq \frac{q_k}{(x_k^*)^{2+\varepsilon}} \qquad \text{for all } j \in D(x^*) \text{ and } k \in R(x^*),$$

and

$$\frac{q_j}{(x_j^*)^{2+\varepsilon}} = \frac{q_k}{(x_k^*)^{2+\varepsilon}} \qquad \text{for all } j,k \in R(x^*).$$

Denoting this latter constant by λ_0, and setting

$$\lambda_i = \lambda_0 - \frac{q_i}{(x_i^*)^{2+\varepsilon}} \qquad \text{for } i = 1,\ldots,N+1,$$

we see that $\lambda_0, \ldots, \lambda_{N+1}$ are non negative and satisfy the Karush-Kuhn-Tucker conditions for x^*, hence x^* is optimal. □

Solution via Hölder's inequality:

Theorem 1 (Hölder's inequality) *$\forall p,q$ with $1 < q,p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $\forall a,b \in \mathbb{R}^n$:*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}$$

With equality if and only if a^p and b^q are linearly dependent, meaning that there exist real numbers $\alpha, \beta \geq 0$, not both of them zero, such that

$$\alpha a_i^p = \beta b_i^q \quad \forall i \in \{1,\ldots,n\}$$

Proof without

Proof First the reader may verify that a minimal configuration of the x_i always satisfies $\sum_{i=1}^{N+1} x_i \leq r$ with equality, because if there is a minimal solution with $\sum_{i=1}^{N+1} x_i < r$ there is always a better one with $\sum_{i=1}^{N+1} x_i = r$. Thus we only have to determine the minimal configuration of $\sum_{i=1}^{N+1} \frac{q_i}{{x_i}^{1+\varepsilon}}$ with $\sum_{i=1}^{N+1} x_i = r$.

We assume at first that the found optimal solution already satisfies the condition $x_i \geq d \quad \forall i \in \{1, \ldots, N+1\}$, that is $D(x) = \emptyset$.

Set $p := \frac{2+\varepsilon}{1+\varepsilon} \geq 0$, $q := 2+\varepsilon \geq 0$. This satisifies the condition for p and q, we verify: $\frac{1}{p} + \frac{1}{q} = \frac{1+\varepsilon}{2+\varepsilon} + \frac{1}{2+\varepsilon} = 1$. We also choose:

$$\begin{aligned} a_i &:= x^{\frac{1}{p}} \\ b_i &:= \left(\frac{q_i}{{x_i}^{1+\varepsilon}}\right)^{\frac{1}{q}} \end{aligned}$$

This yields:

$$\begin{aligned} &\sum_i a_i b_i = \sum_i x^{\frac{1}{p}} \left(\frac{q_i}{{x_i}^{1+\varepsilon}}\right)^{\frac{1}{q}} = \sum_i x_i^{\frac{1}{p} - \frac{1+\varepsilon}{q}} q_i^{\frac{1}{q}} = \sum_i q_i^{\frac{1}{2+\varepsilon}} \\ &\quad \text{with } \frac{1}{p} - \frac{1+\varepsilon}{q} = \frac{2+\varepsilon}{1+\varepsilon} - \frac{2+\varepsilon}{1+\varepsilon} = 0 \end{aligned}$$

We expand the other sums, too:

$$\begin{aligned} \sum_i a_i^p &= \sum_i (x_i^{\frac{1}{p}})^p = \sum_i x_i = r \\ \sum_i b_i^q &= \sum_i \left(\frac{q_i}{{x_i}^{1+\varepsilon}}\right)^{\frac{1}{q}q} = \sum_i \frac{q_i}{{x_i}^{1+\varepsilon}} \end{aligned}$$

Using Hölder's Inequality we calculate:

$$\begin{aligned} \sum_{i=1}^{n} a_i b_i &\leq (\sum_{i=1}^{n} a_i^p)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \\ \Leftrightarrow \sum_i q_i^{\frac{1}{2+\varepsilon}} &\leq r^{\frac{1}{p}} \left(\sum_i \frac{q_i}{{x_i}^{1+\varepsilon}}\right)^{\frac{1}{q}} \\ \Leftrightarrow \sum_i q_i^{\frac{1}{2+\varepsilon}} &\leq r^{\frac{1+\varepsilon}{2+\varepsilon}} \left(\sum_i \frac{q_i}{{x_i}^{1+\varepsilon}}\right)^{\frac{1}{2+\varepsilon}} \\ \Leftrightarrow \frac{\left(\sum_i q_i^{\frac{1}{2+\varepsilon}}\right)^{2+\varepsilon}}{r^{1+\varepsilon}} &\leq \sum_i \frac{q_i}{{x_i}^{1+\varepsilon}} \end{aligned}$$

That is, we have found the lower bound for a minimal solution.

Using the equality case of Hölder's Inequality we obtain the minimal configuration that satisfies the lower bound above with equality:

$$\begin{aligned}
\forall i \in \{1,\dots,n\} : \alpha a_i^p &= \beta b_i^q \\
\overset{\lambda:=\frac{\beta}{\alpha}}{\Leftrightarrow} a_i^p &= \lambda b_i^q \\
\Leftrightarrow x_i^{\frac{1}{p}p} &= x_i = \lambda \left(\frac{q_i}{{x_i}^{1+\varepsilon}} \right)^{\frac{1}{q}q} \\
\Leftrightarrow x_i &= \lambda \frac{q_i}{{x_i}^{(1+\varepsilon)}} \\
\Leftrightarrow x_i^{2+\varepsilon} &= \lambda q_i \\
\Leftrightarrow x_i &= (\lambda q_i)^{\frac{1}{2+\varepsilon}} \\
\Leftrightarrow x_i &= \lambda^{\frac{1}{2+\varepsilon}} q_i^{\frac{1}{2+\varepsilon}}
\end{aligned}$$

We already know that $\sum_{i=1}^{N+1} x_i = r$ and with that we solve for λ:

$$\begin{aligned}
\sum_i (\lambda q_i)^{\frac{1}{2+\varepsilon}} &= \lambda^{\frac{1}{2+\varepsilon}} \sum_k q_k^{\frac{1}{2+\varepsilon}} = r \\
\Leftrightarrow \lambda^{\frac{1}{2+\varepsilon}} &= \frac{r}{\sum_k (q_k)^{\frac{1}{2+\varepsilon}}}
\end{aligned}$$

Applying this we find the optimal x_i that satisfy the lower bound determined by Hölder's Inequality with equality:

$$x_i = \lambda^{\frac{1}{2+\varepsilon}} q_i^{\frac{1}{2+\varepsilon}} = r \frac{\sqrt[2+\varepsilon]{q_i}}{\sum_i \sqrt[2+\varepsilon]{q_i}}$$

As it can be seen by the deduction above there exists only a single solution for λ: we can assume that $\alpha \neq 0$, that is $b \neq 0$, because we can safely assert that $\exists i : q_i \neq 0$. Otherwise the problem would be degenerate and all valid solutions are optimal.

So far we have only looked at the subproblem with $D(x) = \emptyset$, so generalizing to $D(x) \neq \emptyset$, we need to solve the problem:

$$\min \sum_{i=1}^{N+1} \frac{q_i}{{x_i}^{1+\varepsilon}} \Leftrightarrow \min \sum_{i \in R(x)} \frac{q_i}{{x_i}^{1+\varepsilon}} + \sum_{i \in D(x)} \frac{q_i}{d^{1+\varepsilon}}$$

$$\text{with} \sum_{i=1}^{N+1} x_i = \sum_{i \in R(x)} x_i + \sum_{i \in D(x)} x_i = \sum_{i \in R(x)} x_i + |D(x)|\, d = r$$

Thus the problem is equivalent to solving the special problem above for indices in $R(x)$ with the condition $\sum_{i \in R(x)} x_i = r - |D(x)|\, d$ resulting in the optimal solution:

$$x_i = (r - |D(x)|\, d) \frac{\sqrt[2+\varepsilon]{q_i}}{\sum_{k \in R(x)} \sqrt[2+\varepsilon]{q_k}}$$

for $i \in R(x)$.

Algorithm Solving The Optimal Wire Spacing
After proofing Lemma 1 there only remains the determination of the set D. By ordering the q_i's in an increasing row $q_1 \leq ... \leq q_{N+1}$ first, we get an ordered solution vector x'^*, with $x_1^* \leq ... \leq x_{N+1}^*$. The algorithm below takes the following steps to solve the optimal wire spacing for a given instance:

- order q_i's $\Rightarrow q_1 \leq ... \leq q_{N+1}$
- begin with $D = \emptyset$ and compute $x_1'^*$ like in Lemma 1
  - $x_1'^* > d$, compute all further x_i' for $i \in \{2, ..., N+1\}$
  - $x_1'^* > d$, set $x_1^* = d, D = \{1\}$ and compute x_2^*
- permute back x'^* to the original ordering x^*

Algorithm 1 Solving the Wire Spacing Problem

```
Input: An instance $(N, q_1, \ldots, q_{N+1}, r, d, \varepsilon)$ of OWP with $r > (N+1)d$ and $0 < q_1, \ldots, q_{N+1}, \varepsilon$.
Output: An optimal solution $x^*$.
Sort $(q_1, \ldots, q_{N+1})$ to obtain $(q'_1, \ldots, q'_{N+1})$ with $q'_1 \leq \ldots \leq q'_{N+1}$.
$S \leftarrow \sum_{i=1}^{N+1} \sqrt[2+\varepsilon]{q'_i}$ and $\Delta \leftarrow 0$.
for $i = 1, \ldots N+1$ do
    $x'_i \leftarrow \sqrt[2+\varepsilon]{q'_i} \cdot S^{-1} (r - \Delta \cdot d)$
    if $x'_i \leq d$ then
        $x'_i \leftarrow d$, $S \leftarrow S - \sqrt[2+\varepsilon]{q'_i}$, $\Delta \leftarrow \Delta + 1$
    end if
end for
Permute back $x'$ to obtain $x^*$, so that $x^*$ corresponds to the original order of $(q_1, \ldots, q_{N+1})$.
```

Theorem 2 *Algorithm 1 is correct and requires at most* $\mathcal{O}(N \log N)$ *arithmetic operations in the real RAM model of computation.*

Proof For ease of notation we assume that $q'_1 = q_1, \ldots, q'_{N+1} = q_{N+1}$ i.e., the $q_1, \ldots, q_{N+1}$ are already sorted and thus $x' = x^*$ in the algorithm. Let $S^{(i)}$ and $\Delta^{(i)}$ denote the values of S and Δ after the i-th pass through the 'for' loop. Denote by x^* the solution produced by the algorithm and let $S^* = S^{(N+1)}$, $\Delta^* = \Delta^{(N+1)}$ be the final values of S and Δ, respectively. We show that condition (1) of Lemma 1 holds for all $j \in D(x^*)$ and $k \in R(x^*)$.

The first inequality of (1) is clear. Let $m = \max\{l : l \in D(x^*)\}$, then $S^* = S^{(m)}$ and $\Delta^* = \Delta^{(m)} = m$. Monotonicity of $q_1, \ldots, q_{N+1}$ implies that $x_1^* \leq \ldots \leq x_{N+1}^*$, thereby establishing the equality part of (1) for all $k \in R(x^*)$. To prove the second inequality of (1), first note that

$$x_j^* = d, \quad x_k^* = \frac{\sqrt[2+\varepsilon]{q_k}}{S^*} (r - \Delta^* \cdot d).$$

Then the fact that $S^* = S^{(m-1)} - \sqrt[2+\varepsilon]{q_m}$ and $\Delta^* = (m-1)+1 = \Delta^{(m-1)}+1$ implies

$$
\begin{aligned}
x_k^* &= \frac{\sqrt[2+\varepsilon]{q_k}}{\sqrt[2+\varepsilon]{q_j}} \frac{\sqrt[2+\varepsilon]{q_j}}{S^*}(r - \Delta^* \cdot d) \leq \frac{\sqrt[2+\varepsilon]{q_k}}{\sqrt[2+\varepsilon]{q_j}} \frac{\sqrt[2+\varepsilon]{q_m}(r - \Delta^* \cdot d)}{S^*} \\
&= \frac{\sqrt[2+\varepsilon]{q_k}}{\sqrt[2+\varepsilon]{q_j}} \left(\frac{\sqrt[2+\varepsilon]{q_m}(r - \Delta^{(m-1)} \cdot d)}{S^{(m-1)} - \sqrt[2+\varepsilon]{q_m}} - \frac{d \sqrt[2+\varepsilon]{q_m}}{S^{(m-1)} - \sqrt[2+\varepsilon]{q_m}} \right) \\
&= \frac{\sqrt[2+\varepsilon]{q_k}}{\sqrt[2+\varepsilon]{q_j}} \left(\frac{\sqrt[2+\varepsilon]{q_m}(r - \Delta^{(m-1)} \cdot d) \cdot S^{(m-1)}}{S^{(m-1)} \cdot \left(S^{(m-1)} - \sqrt[2+\varepsilon]{q_m}\right)} - \frac{d \sqrt[2+\varepsilon]{q_m}}{S^{(m-1)} - \sqrt[2+\varepsilon]{q_m}} \right).
\end{aligned}
$$

Since

$$
\frac{\sqrt[2+\varepsilon]{q_m}(r - \Delta^{(m-1)} \cdot d)}{S^{(m-1)}} \leq d
$$

we conclude

$$
x_k^* \leq \frac{\sqrt[2+\varepsilon]{q_k}}{\sqrt[2+\varepsilon]{q_j}} d \left(\frac{S^{(m-1)}}{S^{(m-1)} - \sqrt[2+\varepsilon]{q_m}} - \frac{\sqrt[2+\varepsilon]{q_m}}{S^{(m-1)} - \sqrt[2+\varepsilon]{q_m}} \right) = \frac{\sqrt[2+\varepsilon]{q_k}}{\sqrt[2+\varepsilon]{q_j}} d.
$$

Hence, by Lemma 1, x^* is the optimal solution.

The sorting step requires at most $\mathcal{O}(N \log N)$ arithmetic operations. The 'for' loop is executed $N + 1$ times and each passage requires a constant number of arithmetic operations. Therefore the algorithm requires at most $\mathcal{O}(N \log N)$ arithmetic operations. □

5 Optimal Wire Ordering: Field Ⓐ and Ⓑ

Let us take a look at how to achieve the optimal ordering of the wires, that is the problem how to find a permutation π of the indices that minimizes

$$\sum_{i\in D^\pi} \frac{s_{\pi(i-1)}+s_{\pi(i)}}{d} + \frac{1}{\left(r-|D^\pi|\,d\right)^{1+\varepsilon}}\left(\sum_{i\in R^\pi} \sqrt[2+\varepsilon]{s_{\pi(i-1)}+s_{\pi(i)}}\right)^{2+\varepsilon}$$

[1] shows that for $\varepsilon = 0$ the optimal wire ordering is achieved using an ordering that can be best described by a picture:

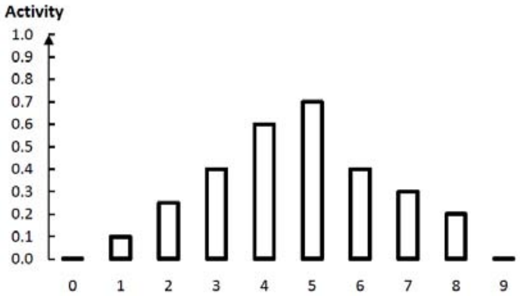

Fig. 4 Optimal Wire Ordering

Thus if τ is a permutation with $s_{\tau(0)} \leq s_{\tau(1)} \leq s_{\tau(2)} \leq \ldots \leq s_{\tau(N)} \leq s_{\tau(N+1)}$, then we can define π as:

$$\pi(k) := \begin{cases} 2k & : 0 \leq k \leq \frac{N+1}{2} \\ 2(N-k)+3 & : \frac{N}{2}+1 \leq k \leq N+1 \end{cases}$$

The same ordering is optimal for the general case, too. The original proof uses a theorem by Supnick and some helper lemmas to show that it is indeed optimal. We only want to describe for the interested reader who has read [1] how to modify that proof to be correct for any $\varepsilon \geq 0$.

In [2] it is shown that if the distances of a TSP can be written as a Supnick matrix the problem is solvable using Supnick's theorems. A Supnick matrix is a symmetric matrix $D = (d_{i,j})_{i,j\in[n]}$ for which the Monge property holds:

$$d_{i,j} + d_{r,s} \leq d_{i,s} + d_{r,j} \quad \forall i,j,r,s : 1 \leq j < r \leq n \text{ and } 1 \leq j < s \leq n$$

It immediately follows from the original proof (because of monotony) that if the Monge property holds for $d_{i,j} := \sqrt{s_i + s_j}$, it also holds for $d_{i,j} := \sqrt[2+\varepsilon]{s_i + s_j}$. It's obvious that D is symmetric. Consequently it's a Supnick matrix and thus the orginal argument is valid. One also easily verifies that the remaining proof can be adapted to using $x_i^{1+\varepsilon}$ instead of x_i without loss of correctness.

6 Optimal Wire Spacing: Field Ⓐ,Ⓑ and Ⓒ

Problem: Optimal Wire Spacing (OWS)
Instance: $N \in \mathbb{N}$; $q_1, \ldots, q_{N+1}, z_1, \ldots, z_{N+1} \in [0, \infty[$; $d, r \in]0, \infty[$.
Question: Decide whether there exists a solution of

$$\begin{aligned} \min \sum_{i=1}^{N+1} \frac{q_i}{{x_i}^{1+\varepsilon}} &+ \frac{z_i}{x_i + x_{i+1}} \\ \text{s. th.} \qquad & \sum_{i=1}^{N+1} x_i \leq r \\ & x_i \geq d \qquad (i = 1, \ldots, N+1), \end{aligned}$$

and, if so, give one.

The z_i are given in analogy to the q_i. To determine a feasible Minimum we now have a closer look at the objective function.

Proof of convexity: Compute the partial differentiation:

$$\frac{\partial F}{\partial x_i} = -(1+\varepsilon)\frac{q_i}{x_i^{2+\varepsilon}} - \frac{z_i}{(x_i + x_{i+1})^2}$$

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \begin{cases} (1+\varepsilon)(2+\varepsilon)\dfrac{q_i}{x_i^{3+\varepsilon}} + \dfrac{2z_i}{(x_i + x_{i+1})^3} & \text{for } i = j; \\ \frac{2z_i}{(x_i + x_{i+1})^3} & \text{for } j = i+1; \\ 0 & \text{else.} \end{cases}$$

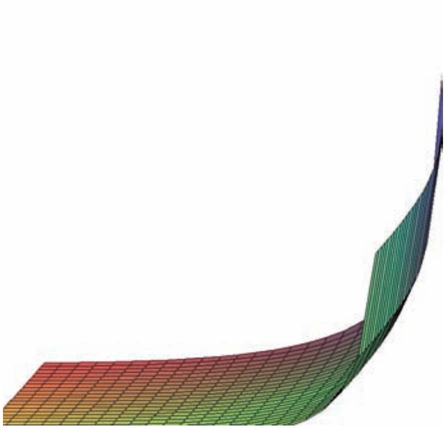

Fig. 5 Objective function F(x)

Due to the fact that the Hessian Matrix is positive definite we can infer that F(x) is strictly convex on the feasible region and its minimum is unique.

By the Karush-Kuhn-Tucker-Theorem a feasible vektor $x^* = (x_1^*, \dots, x_{N+1}^*)^T$ is optimal only if there exist non negative Lagrange multipliers $\lambda_0, \dots, \lambda_{N+1}$ such that

$$\begin{aligned} (1+\varepsilon)\frac{q_i}{x_i^{2+\varepsilon}} - \frac{z_i}{(x_i + x_{i+1})^2} = -\nabla F(x^*)^T u_i &= \lambda_0 - \lambda_i \quad (i = 1, \dots, N+1) \\ \lambda_0 \left(r - \mathbb{1}^T x^*\right) &= 0 \\ \lambda_i \left(x_i^* - d\right) &= 0 \qquad\qquad (i = 1, \dots, N+1) \end{aligned}$$

Let x^* be the optimal solution. Since all q_i's are positive, we have $\lambda_0 > 0$, and hence

$$\sum_{i=1}^{N+1} x_i^* = r \tag{2}$$

As a consequence, $r > (N+1)d$ implies that $R(x^*) \neq \emptyset$.

Now, let $i, k \in R(x^*)$. Then $x_i^*, x_k^* > d$, hence $\lambda_i = \lambda_k = 0$ and therefore

$$\begin{aligned} 0 = \lambda_i &= \lambda_0 - (1+\varepsilon)\frac{q_i}{x_i^{*(2+\varepsilon)}} + \frac{z_i}{(x_i^* + x_{i+1}^*)^2} \\ &= \lambda_0 - (1+\varepsilon)\frac{q_k}{x_k^{*(2+\varepsilon)}} + \frac{z_k}{(x_k^* + x_{k+1}^*)^2} = \lambda_k \end{aligned}$$

$$\Leftrightarrow (1+\varepsilon)\frac{q_i}{x_i^{2+\varepsilon}} - \frac{z_i}{(x_i + x_{i+1})^2} = (1+\varepsilon)\frac{q_k}{x_k^{2+\varepsilon}} - \frac{z_k}{(x_k + x_{k+1})^2}$$

By solving the equation for x_{i+1} one achieves a recursion for x_i, where x_k and x_{k+1} stay constant. Let $x_1 = a$ be a constant with $d \leq a \leq r - |D(x*)|d$. Now we can compute all x_i subject to x_k^* and $x_{k+1}^* := x_i^*(x_k, x_{k+1})$.

With (2):

$$r = \sum_{i=1}^{N+1} x_i^* = \sum_{i \in D(x^*)} x_i^* + \sum_{i \in R(x^*)} x_i^* = |D(x^*)|d + \sum_{i \in R(x^*)} x_i^*(x_k, x_{k+1})$$

As $x_i^*(x_k, x_{k+1})$ is a function that depends on two variables, we need to fix an initial value for x_k^*.
Let $x_1^* = b$ be a constant with $d \leq b \leq r - |D(x*)|d$. Now we get iteratively the optimal values for all x_k^*, $k \in R(x*)$. These values are feasible as well as numerically computable but the exact solution term is too complicated to be specified here. □

7 Optimal Wire Spacing With Probabilities

In this paragraph we treat the cases that the wires switch concurrent or contrary. In our first model we consider three cases and define the following variable:

$$\hat{c}\,(i,j) := \begin{cases} +1 & \text{wire i and j switch concurrent} \\ 0 & \text{no information about wire i and j} \\ -1 & \text{wire i and j switch contrary} \end{cases}$$

The capacitance and losses will change as follows: For $\hat{c}\,(i,j) = 1$ there is no field between wire i and j and so there is no loss. For $\hat{c}\,(i,j) = -1$ accrues double capacitance and so double loss. For $\hat{c}\,(i,j) = 0$ we consider the normal field.
By define $c\,(i,j) := 1 - \hat{c}\,(i,j)$ the loss between 2 neighbored wires is proportional to $c\,(i,i+1)\,\frac{1}{x_i}$.
The objective function of our optimization problem will change to:

$$\min \sum_{i=1}^{N} s_{\pi(i)} \left(\frac{c\,(\pi(i-1),\pi(i))}{x_{i-1}} + \frac{c\,(\pi(i),\pi(i+1))}{x_i} \right)$$

The equation can be tranformed to:

$$\min \sum_{i=1}^{N+1} \frac{s\,(\pi(i-1)) + s\,(\pi(i))}{x_i} c\,(\pi(i-1),\pi(i))$$

With $\hat{q}_i := \underbrace{s\,(\pi(i-1)) + s\,(\pi(i))}_{\geq 0} \cdot \underbrace{c\,(\pi(i-1),\pi(i))}_{\geq 0} \geq 0$ we get the optimzation problem $\min \sum_{i=1}^{N+1} \frac{\hat{q}_i}{x_i}$ with our standard constraints.
Because of $\hat{q}_i \geq 0$ we can use Lemma 1 to find a OWS solution.

In the second model we consider that the wires switch contrary with a probability p. So we can define our $c(i,j) := 2p$ and we can use again our OWS algorithm.

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